The Ratio of (Independent) Exponential Random Variables

Steven W. Nydick

May 25, 2012

Say we have two, indepenent, exponentially distributed variables X and Y, both with rate parameter λ. Then we can write the distributions of X and Y as

$$f_X(x) = \lambda \exp[-\lambda x] \qquad\qquad f_Y(y) = \lambda \exp[-\lambda y].$$

And because X and Y are independent, the joint distribution is the product of the marginal distributions, or

$$f_{X,Y}(x, y) = f_X(x) \times f_Y(y) = \lambda \exp[-\lambda x] \times \lambda \exp[-\lambda y] = \lambda^2 \exp[-\lambda(x + y)]. \tag{1}$$

Our goal is to determine the distribution of $\frac{X}{Y}$. But because $f_{X,Y}(x, y)$ is a bivariate probability density function (albeit one that can be factored into the product of its marginals), we must define two functions of X and Y, U and V, so that the mapping of $(X, Y) \to (U, V)$ is one-to-one. Noting that the sum of independent exponentially distributed variables is an easily recognizable distribution, let

$$U = \frac{X}{Y}$$
$$V = X + Y$$

so that the inverse mapping $U = \frac{X}{Y} \implies Y = \frac{X}{U}$, and

$$V = X + Y = X + \frac{X}{U} = X\left(1 + \frac{1}{U}\right)$$

which implies that

$$X = \frac{V}{1 + \frac{1}{U}} = \frac{UV}{U + 1} \tag{2}$$
$$Y = \frac{X}{U} = \frac{V}{U + 1}. \tag{3}$$

Because $(X, Y) \to (U, V)$ is invertible, we can apply the change-of-variables formula.

Given an invertible map from $(X, Y) \to (U, V)$, the change of variables formula implies

$$f_{U,V}(u, v) = \left|J\big(x(u, v), y(u, v)\big)\right| f_{X,Y}(u, v) \tag{4}$$

where $\left|J\big(x(u,v),y(u,v)\big)\right|$ is the Jacobian of the transformation from (U,V) back to (X,Y). In our case, the Jacobian matrix is just a 2×2 matrix of partial derivatives, or

$$\left|J\big(x(u,v),y(u,v)\big)\right| = \begin{vmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{(u+1)^2} & \frac{u}{u+1} \\ \frac{-v}{(u+1)^2} & \frac{1}{u+1} \end{vmatrix} = \frac{v}{(u+1)^3} + \frac{uv}{(u+1)^3} = \frac{v}{(u+1)^2}. \tag{5}$$

We finally need to use the above equations to solve for $f_{U,V}(u,v)$. Plugging Equations (5) (the Jacobian part), (1) (the density part), and (2) and (3) (the change from $(X,Y)\to(U,V)$ part) into Equation (4) (the change-of-variables formula), we obtain

$$\begin{aligned} f_{U,V}(u,v) = \left|J\big(x(u,v),y(u,v)\big)\right| f_{X,Y}(u,v) &= \frac{v}{(u+1)^2}\lambda^2 \exp\left[-\lambda\left(\frac{uv}{u+1}+\frac{v}{u+1}\right)\right] \\ &= \frac{v}{(u+1)^2}\lambda^2 \exp\left[-\lambda\left(\frac{v(u+1)}{u+1}\right)\right] \\ &= \frac{v}{(u+1)^2}\lambda^2 \exp[-\lambda v] \\ &= \left[\frac{1}{(u+1)^2}\right]\left[\lambda^2 v \exp[-\lambda v]\right] = f_U(u)\times f_V(v). \end{aligned} \tag{6}$$

As shown in Equation (6), $f_{U,V}(u,v) = f_U(u)\times f_V(v)$, so that U and V are also independent random variables. Moreover, recognize that $f_V(v) = \lambda^2 v\exp[-\lambda v]$ is the formula for a Gamma distribution with shape parameter $k=2$ and rate parameter λ (or an Erlang distribution, which is a special case of the Gamma distribution with $k\in\mathbb{N}$). Therefore, no multiplicative constant is needed and

$$f_U(u) = \frac{1}{(u+1)^2} \tag{7}$$

where $u\in(0,\infty)$ because $x\in(0,\infty)$ and $y\in(0,\infty)$, which is what we wanted to demonstrate. ■

One relationship between U and another brand-name distribution is through a simple transformation. Let $Z=\exp[U]$. Then

$$f_Z(z) = \left|\frac{dz}{du}\right| f_U(z) = \exp[z]\frac{1}{(\exp[z]+1)^2} = \frac{\exp[z]}{(1+\exp[z])^2}$$

which is the logistic distribution with location parameter $\mu=0$ and scale parameter $s=1$. One could think of the distribution of U as a log-logistic distribution with $\alpha=\beta=1$, where α and β are the scale and shape parameters respectively. As can easily be seen from the pdf of the random variable U (Equation 7), the expected value of U is undefined.