Euler's Formula and Trig Identities

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Introduction

Many of the identities from trigonometry can be demonstrated relatively easily using Euler's formula, rules of exponents, basic complex analysis, multiplication rules, and *one* elementary rule from trigonometry. There are two things to consider during the remaining derivation. First, Euler's formula says

$$\exp[ix] = \cos(x) + i\sin(x) \tag{1}$$

Equation (1) is due to the consequences of rotating a line in the complex plane. See Wikipedia for a more thorough explanation. Second, the real and imaginary part of a complex number remain the same before and after factoring. No manipulation can turn the imaginary part of a complex number into a real part.

Pythagorean Identities

Before using Euler's formula to demonstrate complex trigonometric identities, one might not remember the most basic trigonometric identity of them all: $\cos^2(x) + \sin^2(x) = 1$. Due to the Pythagorian theorem

$$h^2 = o^2 + a^2. \tag{2}$$

And from the definitions of sine ($\sin(x) = \frac{o}{h}$, where o is the length of the horizontal or vertical line opposite of the angle, x, and h is the hypotenuse formed from a right triangle) and cosine ($\cos(x) = \frac{a}{h}$, where a is the length of the horizontal or vertical line adjacent to the angle, x), we have that

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= \left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 \\
&= \frac{o^2 + a^2}{h^2} \\
&= \frac{o^2 + a^2}{o^2 + a^2} \\
&= 1
\end{aligned} \tag{3}$$

Even/Odd Identities

Using Equation (3), we can figure out some relatively basic identities. For instance, $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ are both solved in one step.

$$\begin{aligned}
\cos(-x) + i\sin(-x) = \exp[-ix] &= (\exp[ix])^{-1} && (4)\\
&= \frac{1}{\exp(ix)} \\
&= \frac{1}{\cos(x) + i\sin(x)} && \text{(by 1)}\\
&= \frac{1}{\cos(x) + i\sin(x)} \left(\frac{\cos(x) - i\sin(x)}{\cos(x) - i\sin(x)}\right) && (5)\\
&= \frac{\cos(x) - i\sin(x)}{\cos^2(x) + \sin^2(x)} && \text{(using } i^2 = -1\text{)}\\
&= \frac{\cos(x) - i\sin(x)}{1} && \text{(by 3)}\\
&= \cos(x) + i[-\sin(x)]
\end{aligned}$$

And because the real/imaginary part of $\cos(-x) + i\sin(-x)$ is equal to the real/imaginary part of $\cos(x) + i[-\sin(x)]$, we end up with

$$\cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x) \tag{6}$$

Many trig identities use Equation (6) in their derivation.

Sum/Difference Formulas

A straightforward derivation involves sums or differences inside sines and cosines. The derivation proceeds by noting the relationship among angles and Euler's formula

$$\cos(x \pm y) + i\sin(x \pm y) = \exp[i(x \pm y)].$$

Using properties of exponents

$$\begin{aligned}
\exp[i(x \pm y)] &= \exp[ix]\exp[i(\pm y)] \\
&= [\cos(x) + i\sin(x)][\cos(\pm y) + i\sin(\pm y)] \\
&= [\cos(x) + i\sin(x)][\cos(y) \pm i\sin(y)] && \text{(by 6)}\\
&= \cos(x)\cos(y) \mp \sin(x)\sin(y) + i[\sin(x)\cos(y) \pm \cos(x)\sin(y)]
\end{aligned}$$

And because the real/imaginary part of $\cos(x \pm y) + i\sin(x \pm y)$ is equal to the real/imaginary part of $\cos(x)\cos(y) \mp \sin(x)\sin(y) + i[\sin(x)\cos(y) \pm \cos(x)\sin(y)]$, we end up with

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \qquad \sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \tag{7}$$

Double Angle Formulas

$\cos(2x)$ and $\sin(2x)$

Another useful identity entails doubling an angle inside of a sine or consine operator. Even though this identity might not seem important, a consequence of the double angle formula is very useful. This derivation is similar to the previous one and starts by considering that

$$\cos(2x) + i\sin(2x) = \exp[i(2x)].$$

And the derivation follows a similar path to the previous one.

$$\begin{aligned}\exp[i(2x)] = \exp[i(x+x)] &= \exp[ix]\exp[ix] \\ &= (\exp[ix])^2 \\ &= [\cos(x) + i\sin(x)]^2 \\ &= \cos^2(x) + 2i\cos(x)\sin(x) - \sin^2(x) \\ &= \cos^2(x) - \sin^2(x) + i[2\cos(x)\sin(x)]\end{aligned}$$

And because the real/imaginary part of $\cos(2x) + i\sin(2x)$ is equal to the real/imaginary part of $\cos^2(x) - \sin^2(x) + i[2\cos(x)\sin(x)]$, we end up with

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sin(2x) = 2\cos(x)\sin(x) \tag{8}$$

$\cos^2(x)$ and $\sin^2(x)$

Immediate consequences of Equations (3) and (8) include indentites for squares of sines and cosines. For instance, we know that

$$\begin{aligned}\cos(2x) + 1 &= \cos^2(x) - \sin^2(x) + 1 && \text{(by 8)} \\ &= \cos^2(x) - \sin^2(x) + [\sin^2(x) + \cos^2(x)] && \text{(by 3)} \\ &= 2\cos^2(x)\end{aligned}$$

and that

$$\begin{aligned}\cos(2x) - 1 &= \cos^2(x) - \sin^2(x) - 1 && \text{(by 8)} \\ &= \cos^2(x) - \sin^2(x) - [\sin^2(x) + \cos^2(x)] && \text{(by 3)} \\ &= -2\sin^2(x) && (9)\end{aligned}$$

so that we immediately end up with

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \sin^2(x) = \frac{1 - \cos(2x)}{2} \tag{10}$$

Sum to Product Formulas

$\cos(x) - \cos(y)$ and $\sin(x) - \sin(y)$

We can also examine identities involving the differences outside of sines and cosines. For instance, we can put $\cos(x) - \cos(y)$ and $\sin(x) - \sin(y)$ into one step by noting that

$$\cos(x) - \cos(y) + i[\sin(x) - \sin(y)] = \cos(x) + i\sin(x) - [\cos(y) + i\sin(y)] = \exp(ix) - \exp(iy).$$

And therefore

$$\begin{aligned}
\exp(ix) - \exp(iy) &= \exp\left(ix + i\tfrac{y}{2} - i\tfrac{y}{2}\right) - \exp\left(iy + i\tfrac{x}{2} - i\tfrac{x}{2}\right) \\
&= \exp\left(i\tfrac{x+y}{2} + i\tfrac{x-y}{2}\right) - \exp\left(i\tfrac{x+y}{2} + i\tfrac{y-x}{2}\right) \\
&= \exp\left(i\tfrac{x+y}{2}\right)\exp\left(i\tfrac{x-y}{2}\right) - \exp\left(i\tfrac{x+y}{2}\right)\exp\left(i\tfrac{y-x}{2}\right) \\
&= \exp\left(i\tfrac{x+y}{2}\right)\left[\exp\left(i\tfrac{x-y}{2}\right) - \exp\left(i\tfrac{y-x}{2}\right)\right] \\
&= \left[\cos\left(\tfrac{x+y}{2}\right) + i\sin\left(\tfrac{x+y}{2}\right)\right]\left[\cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right) - \cos\left(\tfrac{y-x}{2}\right) - i\sin\left(\tfrac{y-x}{2}\right)\right]
\end{aligned}$$

Using Equation (6), we have

$$\begin{aligned}
&= \left[\cos\left(\tfrac{x+y}{2}\right) + i\sin\left(\tfrac{x+y}{2}\right)\right]\left[\cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right) - \cos\left(\tfrac{y-x}{2}\right) - i\sin\left(\tfrac{y-x}{2}\right)\right] \\
&= \left[\cos\left(\tfrac{x+y}{2}\right) + i\sin\left(\tfrac{x+y}{2}\right)\right]\left[\cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right) - \cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right)\right] \\
&= \left[\cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right)\right]\left[i\left(2\sin\left(\tfrac{x-y}{2}\right)\right)\right] \\
&= -2\sin\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) + i\left[2\sin\left(\tfrac{x-y}{2}\right)\cos\left(\tfrac{x+y}{2}\right)\right]
\end{aligned}$$

And because the real/imaginary part of $\cos(x) - \cos(y) + i[\sin(x) - \sin(y)]$ is equal to the real/imaginary part of $-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) + i\left[2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right]$, we end up with

$$\cos(x) - \cos(y) = -2\sin\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) \qquad \sin(x) - \sin(y) = 2\sin\left(\tfrac{x-y}{2}\right)\cos\left(\tfrac{x+y}{2}\right) \tag{11}$$

$\cos(x) + \cos(y)$ and $\sin(x) + \sin(y)$

A similar derivation indicates involving the sums of sines and cosines. As before, note that

$$\cos(x) + \cos(y) + i[\sin(x) + \sin(y)] = \cos(x) + i\sin(x) + [\cos(y) + i\sin(y)] = \exp(ix) + \exp(iy)$$

And because the negative versus positive does not affect any of the steps inside the exponent, we end up with

$$\begin{aligned}
\exp(ix) - \exp(iy) &= \exp\left(i\tfrac{x+y}{2}\right)\left[\exp\left(i\tfrac{x-y}{2}\right) + \exp\left(i\tfrac{y-x}{2}\right)\right] \\
&= \left[\cos\left(\tfrac{x+y}{2}\right) + i\sin\left(\tfrac{x+y}{2}\right)\right]\left[\cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right) + \cos\left(\tfrac{y-x}{2}\right) + i\sin\left(\tfrac{y-x}{2}\right)\right] \\
&= \left[\cos\left(\tfrac{x+y}{2}\right) + i\sin\left(\tfrac{x+y}{2}\right)\right]\left[\cos\left(\tfrac{x-y}{2}\right) + i\sin\left(\tfrac{x-y}{2}\right) + \cos\left(\tfrac{x-y}{2}\right) - i\sin\left(\tfrac{x-y}{2}\right)\right] \\
&= \left[\cos\left(\tfrac{x+y}{2}\right) + i\sin\left(\tfrac{x+y}{2}\right)\right]\left[2\cos\left(\tfrac{x-y}{2}\right)\right] \\
&= 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) + i\left[2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right)\right]
\end{aligned}$$

And because the real/imaginary part of $\cos(x)+\cos(y)+i[\sin(x)+\sin(y)]$ is equal to the real/imaginary part of $2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+i\left[2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]$, we end up with

$$\cos(x)+\cos(y)=2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) \qquad \sin(x)+\sin(y)=2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) \tag{12}$$

Extensions of the sum-to-product formulas result in product-to-sum formulas. However, because those formulas are not easily derived using Euler's formula, I leave them as exercises.